



Contents lists available at ScienceDirect

Journal of the Egyptian Mathematical Society

journal homepage: www.elsevier.com/locate/joems

Original Article

On the dynamics of a higher order rational difference equations

M.M. El-Dessoky^{a,b}^a King Abdulaziz University, Faculty of Science, Mathematics Department, P. O. Box 80203, Jeddah, 21589, Saudi Arabia^b Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

ARTICLE INFO

Article history:

Received 24 November 2015

Revised 19 January 2016

Accepted 29 June 2016

Available online 20 July 2016

MSC:

39A10

39A11

39A99

34C99

Keywords:

Difference equations

Stability

Global stability

Boundedness

Periodic solutions

ABSTRACT

The main objective of this paper is to study the global stability of the positive solutions and the periodic character of the difference equation

$$y_{n+1} = ay_n + by_{n-t} + cy_{n-l} + \frac{dy_{n-k} + ey_{n-s}}{\alpha y_{n-k} + \beta y_{n-s}}, \quad n = 0, 1, \dots,$$

with positive parameters and non-negative initial conditions. Numerical examples to the difference equation are given to explain our results.

Copyright 2016, Egyptian Mathematical Society. Production and hosting by Elsevier B.V.

This is an open access article under the CC BY-NC-ND license.

(<http://creativecommons.org/licenses/by-nc-nd/4.0/>)

1. Introduction

Difference equations, although their forms look very simple, it is extremely difficult to understand thoroughly the periodic character, the boundedness character and the global behaviors of their solutions. The study of non-linear rational difference equations of higher order is of paramount importance, since we still know so little about such equations. It is worthwhile to point out that although several approaches have been developed for finding the global character of difference equations, relatively a large number of difference equations has not been thoroughly understood yet [1–21].

In recent years non-linear difference equations have attracted the interest of many researchers, for example:

Kalabušić et al. [12] investigated the periodic nature, the boundedness character, and the global asymptotic stability of solutions of the difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_{n-2}}, \quad n = 0, 1, \dots,$$

where the sequence p_n is periodic with period $k_2 = \{2, 3\}$ with positive terms and the initial conditions are positive.

Raafat [15] studied the global attractivity, periodic nature, oscillation and the boundedness of all admissible solutions of the difference equations

$$x_{n+1} = \frac{A - Bx_{n-1}}{\pm C + Dx_{n-2}}, \quad n = 0, 1, \dots,$$

where A, B are non-negative real numbers, C, D are positive real numbers $\pm C + Dx_{n-2} \neq 0$ and for all $n \geq 0$.

Alaa [16] investigated the global stability, the permanence, and the oscillation character of the recursive sequence

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \dots,$$

where α is a negative number and the initial conditions x_{-1} and x_0 are negative numbers.

Obaid et al. [17] investigated the global stability character, boundedness and the periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2} + dx_{n-3}}{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}},$$

where the parameters $a, b, c, d, \alpha, \beta$ and γ are positive real numbers and the initial conditions x_{-3}, x_{-2}, x_{-1} and x_0 are positive real numbers.

In [18] Zayed studied the global stability and the asymptotic properties of the non-negative solutions of the non-linear

E-mail address: dessokym@mans.edu.eg

difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}, \quad n = 0, 1, \dots,$$

where the parameters A, B, p, q and the initial conditions $x_{-k}, \dots, x_{-1}, x_0$ are arbitrary positive real numbers, while k is a positive integer number.

El-Moneam [19] got the periodicity, the boundedness and the global stability of the positive solutions of the non-linear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}},$$

$$n = 0, 1, \dots,$$

where the coefficients $A, B, C, D, b, d, e \in (0, \infty)$, while k, l and σ are positive integers. The initial conditions $x_{-\sigma}, \dots, x_{-l}, \dots, x_{-k}, \dots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < \sigma$.

Our aim in this paper is to study some qualitative behavior of the positive solutions of the difference equation

$$y_{n+1} = ay_n + by_{n-t} + cy_{n-l} + \frac{dy_{n-k} + ey_{n-s}}{\alpha y_{n-k} + \beta y_{n-s}}, \quad n = 0, 1, \dots, \quad (1)$$

where the initial conditions $x_{-\delta}, x_{-\delta+1}, \dots, x_{-1}$ and x_0 are positive real numbers where $\delta = \max\{t, l, k, s\}$ and the coefficients a, b, c, d, e, α and β are positive real numbers.

2. Some basic definition

Let I be some interval of real numbers and let

$$F : I^{\delta+1} \rightarrow I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-\delta}, x_{-\delta+1}, \dots, x_0 \in I$, the difference equation

$$y_{n+1} = F(y_n, y_{n-1}, \dots, y_{n-\delta}), \quad n = 0, 1, \dots, \quad (2)$$

has a unique solution $\{y_n\}_{n=-\delta}^{\infty}$.

Definition 1 (Equilibrium Point). A point $\bar{y} \in I$ is called an equilibrium point of the difference Eq. (2) if

$$\bar{y} = F(\bar{y}, \bar{y}, \dots, \bar{y}).$$

That is, $y_n = \bar{y}$ for $n \geq 0$, is a solution of the difference Eq. (2), or equivalently, \bar{y} is a fixed point of F .

Definition 2 (Stability). Let $\bar{y} \in (0, \infty)$ be an equilibrium point of the difference Eq. (2). Then, we have

- (i) The equilibrium point \bar{y} of the difference Eq. (2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $y_{-\delta}, \dots, y_{-1}, y_0 \in I$ with

$$|y_{-\delta} - \bar{y}| + \dots + |y_{-1} - \bar{y}| + |y_0 - \bar{y}| < \delta,$$

we have

$$|y_n - \bar{y}| < \epsilon \quad \text{for all } n \geq -\delta.$$

- (ii) The equilibrium point \bar{y} of the difference Eq. (2) is called locally asymptotically stable if \bar{y} is locally stable solution of Eq. (2) and there exists $\gamma > 0$, such that for all $y_{-\delta}, \dots, y_{-1}, y_0 \in I$ with

$$|y_{-\delta} - \bar{y}| + \dots + |y_{-1} - \bar{y}| + |y_0 - \bar{y}| < \gamma,$$

we have

$$\lim_{n \rightarrow \infty} y_n = \bar{y}.$$

- (iii) The equilibrium point \bar{y} of the difference Eq. (2) is called global attractor if for all $y_{-\delta}, \dots, y_{-1}, y_0 \in I$, we have

$$\lim_{n \rightarrow \infty} y_n = \bar{y}.$$

- (iv) The equilibrium point \bar{y} of the difference Eq. (2) is called globally asymptotically stable if \bar{y} is locally stable, and \bar{y} is also a global attractor of the difference Eq. (2).

- (v) The equilibrium point \bar{y} of the difference Eq. (2) is called unstable if \bar{y} is not locally stable.

Definition 3 (Periodicity). A sequence $\{y_n\}_{n=-\delta}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -\delta$. A sequence $\{y_n\}_{n=-\delta}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Definition 4. Eq. (2) is called permanent and bounded if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $y_{-\delta}, \dots, y_{-1}, y_0 \in (0, \infty)$ there exists a positive integer N which depends on these initial conditions such that

$$m \leq y_n \leq M \quad \text{for all } n > N.$$

Definition 5. The linearized equation of the difference Eq. (2) about the equilibrium \bar{y} is the linear difference equation

$$x_{n+1} = \sum_{i=0}^{\delta} \frac{\partial F(\bar{y}, \bar{y}, \dots, \bar{y})}{\partial y_{n-i}} x_{n-i}. \quad (3)$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0 \lambda^{\delta} + p_1 \lambda^{\delta-1} + \dots + p_{\delta-1} \lambda + p_{\delta} = 0, \quad (4)$$

where

$$p_i = \frac{\partial F(\bar{y}, \bar{y}, \dots, \bar{y})}{\partial y_{n-i}}.$$

Theorem 1 [3]. Assume that $p_i \in \mathbb{R}$, $i = 1, 2, \dots, \delta$ and δ is non-negative integer. Then

$$\sum_{i=1}^{\delta} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+\delta} + p_1 x_{n+\delta-1} + \dots + p_{\delta} x_n = 0, \quad n = 0, 1, \dots$$

Theorem 2 [4]. Let $g : [\eta, \xi]^{\delta+1} \rightarrow [\eta, \xi]$, be a continuous function, where δ is a positive integer, and where $[\eta, \xi]$ is an interval of real numbers. Consider the difference equation

$$y_{n+1} = g(y_n, y_{n-1}, \dots, y_{n-\delta}), \quad n = 0, 1, \dots \quad (5)$$

Suppose that g satisfies the following conditions.

- (1) For each integer i with $1 \leq i \leq \delta + 1$; the function $g(z_1, z_2, \dots, z_{\delta+1})$ is weakly monotonic in z_i that is if $z_i \geq \hat{z}_i$ then $g(z_1, z_2, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_{\delta+1}) > g(z_1, z_2, \dots, z_{i-1}, \hat{z}_i, z_{i+1}, \dots, z_{\delta+1})$.
- (2) If m, M is a solution of the system

$$m = g(m_1, m_2, \dots, m_{\delta+1}), \quad M = g(M_1, M_2, \dots, M_{\delta+1}),$$

then $m = M$, where for each $i = 1, 2, \dots, \delta + 1$, we set

$$m_i = \begin{cases} m, & \text{if } g \text{ is non-decreasing in } z_i \\ M, & \text{if } g \text{ is non-increasing in } z_i \end{cases}$$

and

$$M_i = \begin{cases} M, & \text{if } g \text{ is non-decreasing in } z_i \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{cases}$$

Then there exists exactly one equilibrium point \bar{y} of Eq. (5), and every solution of Eq. (5) converges to \bar{y} .

3. Local stability

In this section, we study the local stability character of the equilibrium point of Eq. (1).

Eq. (1) has equilibrium point and is given by

$$\bar{y} = a\bar{y} + b\bar{y} + c\bar{y} + \frac{d\bar{y} + e\bar{y}}{\alpha\bar{y} + \beta\bar{y}}, \text{ or } (1 - a - b - c)\bar{y} = \frac{d + e}{\alpha + \beta}$$

if $a + b + c < 1$, then the unique equilibrium point is

$$\bar{y} = \frac{d + e}{(1 - a - b - c)(\alpha + \beta)}.$$

Let $f : (0, \infty)^5 \rightarrow (0, \infty)$ be a continuous function defined by

$$f(v_0, v_1, v_2, v_3, v_4) = av_0 + bv_1 + cv_2 + \frac{dv_3 + ev_4}{\alpha v_3 + \beta v_4}. \quad (6)$$

Therefore, it follows that

$$\begin{aligned} \frac{\partial f(v_0, v_1, v_2, v_3, v_4)}{\partial v_0} &= a, \quad \frac{\partial f(v_0, v_1, v_2, v_3, v_4)}{\partial v_1} = b, \\ \frac{\partial f(v_0, v_1, v_2, v_3, v_4)}{\partial v_2} &= c, \\ \frac{\partial f(v_0, v_1, v_2, v_3, v_4)}{\partial v_3} &= \frac{(d\beta - \alpha e)v_4}{(\alpha v_3 + \beta v_4)^2}, \quad \frac{\partial f(v_0, v_1, v_2, v_3, v_4)}{\partial v_4} \\ &= \frac{(\alpha e - d\beta)v_3}{(\alpha v_3 + \beta v_4)^2}. \end{aligned}$$

Then, we see that

$$\begin{aligned} \frac{\partial f(\bar{y}, \bar{y}, \bar{y}, \bar{y}, \bar{y})}{\partial v_0} &= a = p_1, \quad \frac{\partial f(\bar{y}, \bar{y}, \bar{y}, \bar{y}, \bar{y})}{\partial v_1} = b = p_2, \\ \frac{\partial f(\bar{y}, \bar{y}, \bar{y}, \bar{y}, \bar{y})}{\partial v_2} &= c = p_3, \\ \frac{\partial f(\bar{y}, \bar{y}, \bar{y}, \bar{y}, \bar{y})}{\partial v_3} &= \left(\frac{d\beta - \alpha e}{(\alpha + \beta)^2} \right) \left(\frac{(1 - a - b - c)(\alpha + \beta)}{d + e} \right) \\ &= \frac{(d\beta - \alpha e)(1 - a - b - c)}{(\alpha + \beta)(d + e)} = p_4, \\ \frac{\partial f(\bar{y}, \bar{y}, \bar{y}, \bar{y}, \bar{y})}{\partial v_4} &= \left(\frac{\alpha e - d\beta}{(\alpha + \beta)^2} \right) \left(\frac{(1 - a - b - c)(\alpha + \beta)}{d + e} \right) \\ &= \frac{(\alpha e - d\beta)(1 - a - b - c)}{(\alpha + \beta)(d + e)} = p_5. \end{aligned} \quad (7)$$

The linearized equation of Eq. (1) about \bar{y} , is

$$y_{n+1} = p_1 y_n + p_2 y_{n-1} + p_3 y_{n-2} + p_4 y_{n-3} + p_5 y_{n-4}.$$

Theorem 3. Assume that $d\beta \neq \alpha e$, $1 > a + b + c$ and

$$2|(d\beta - \alpha e)| < (\alpha + \beta)(d + e) \quad (8)$$

then the equilibrium point \bar{y} of Eq. (1) is locally asymptotically stable.

Proof. From (7) and (8) we deduce that

$$|p_1| + |p_2| + |p_3| + |p_4| + |p_5| < 1$$

$$\begin{aligned} |a| + |b| + |c| + \left| \frac{(d\beta - \alpha e)(1 - a - b - c)}{(\alpha + \beta)(d + e)} \right| \\ + \left| \frac{(\alpha e - d\beta)(1 - a - b - c)}{(\alpha + \beta)(d + e)} \right| < 1, \end{aligned}$$

$$\frac{2(1 - a - b - c)}{(\alpha + \beta)(d + e)} |(d\beta - \alpha e)| < 1 - a - b - c.$$

If $1 - a - b - c > 0$, then

$$2|(d\beta - \alpha e)| < (\alpha + \beta)(d + e).$$

The proof is complete. \square

4. Global stability

In this section we study the global stability of the positive solutions of Eq. (1).

Theorem 4. The equilibrium point \bar{y} is a global attractor of Eq. (1) if one of the following conditions holds:

- (i) $d\beta - \alpha e \geq 0$, $a + b + c < 1$.
- (ii) $\alpha e - d\beta \geq 0$, $a + b + c < 1$.

Proof. Let r and s be non-negative real numbers and assume that $g : [r, s]^5 \rightarrow [r, s]$ be a function defined by

$$g(v_0, v_1, v_2, v_3, v_4) = av_0 + bv_1 + cv_2 + \frac{dv_3 + ev_4}{\alpha v_3 + \beta v_4}.$$

Then

$$\begin{aligned} \frac{\partial g(v_0, v_1, v_2, v_3, v_4)}{\partial v_0} &= a, \quad \frac{\partial g(v_0, v_1, v_2, v_3, v_4)}{\partial v_1} = b, \\ \frac{\partial g(v_0, v_1, v_2, v_3, v_4)}{\partial v_2} &= c, \\ \frac{\partial g(v_0, v_1, v_2, v_3, v_4)}{\partial v_3} &= \frac{(d\beta - \alpha e)v_4}{(\alpha v_3 + \beta v_4)^2}, \quad \frac{\partial g(v_0, v_1, v_2, v_3, v_4)}{\partial v_4} \\ &= \frac{(\alpha e - d\beta)v_3}{(\alpha v_3 + \beta v_4)^2}. \end{aligned}$$

We consider two cases:

Case 1: Let $d\beta - \alpha e > 0$, $a + b + c < 1$, and $e - d \neq 0$ is true, then we can easily see that the function $g(v_0, v_1, v_2, v_3, v_4)$ is increasing in v_0, v_1, v_2, v_3 and decreasing in v_4 . Suppose that (m, M) is a solution of the system

$$M = h(M, M, M, M, m) \quad \text{and} \quad m = h(m, m, m, m, M).$$

Then from Eq. (1), we see that

$$\begin{aligned} M &= aM + bM + cM + \frac{dM + em}{\alpha M + \beta m} \quad \text{and} \quad m = am + bm + cm \\ &\quad + \frac{dm + eM}{\alpha m + \beta M} \end{aligned}$$

then

$$\begin{aligned} \alpha(1 - a - b - c)M^2 + \beta(1 - a - b - c)mM &= dM + em, \\ \alpha(1 - a - b - c)m^2 + \beta(1 - a - b - c)Mm &= dm + eM. \end{aligned}$$

Subtracting this two equations, we obtain

$$(M - m)[\alpha(1 - a - b - c)(M + m) + (e - d)] = 0,$$

under the condition $a + b + c \neq 1$ and $e \neq d$ we see that $M = m$. It follows from Theorem 2 that \bar{y} is a global attractor of Eq. (1).

Case 2: Let $\alpha e - d\beta > 0$, $a + b + c < 1$, and $(\alpha - \beta)(1 - a - b - c) \neq d - e$ is true, then we can easily see that the function $g(v_0, v_1, v_2, v_3, v_4)$ is increasing in v_0, v_1, v_2, v_4 and decreasing in v_3 . Suppose that (m, M) is a solution of the system

$$M = h(M, M, M, m, M) \quad \text{and} \quad m = h(m, m, m, M, m).$$

Then from Eq. (1), we see that

$$\begin{aligned} M &= aM + bM + cM + \frac{dm + eM}{\alpha m + \beta M} \quad \text{and} \quad m = am + bm + cm \\ &\quad + \frac{dM + em}{\alpha M + \beta m} \end{aligned}$$

then

$$\begin{aligned} \alpha(1 - a - b - c)Mm + \beta(1 - a - b - c)M^2 &= dm + eM, \\ \alpha(1 - a - b - c)mM + \beta(1 - a - b - c)m^2 &= dM + em. \end{aligned}$$

Subtracting this two equations, we obtain

$$(M - m)[\beta(1 - a - b - c)(M + m) + (d - e)] = 0,$$

under the condition $a + b + c \neq 1$ and $e \neq d$ we see that $M = m$. It follows from Theorem 2 that \bar{y} is a global attractor of Eq. (1). This completes the proof. \square

5. Boundedness of solutions

In this section we investigate the boundedness nature of the solutions of Eq. (1).

Theorem 5. Every solution of Eq. (1) is bounded if $a + b + c < 1$.

Proof. Let $\{y_n\}_{n=-\delta}^{\infty}$ be a solution of Eq. (1). It follows from Eq. (1) that

$$\begin{aligned} y_{n+1} &= ay_n + by_{n-t} + cy_{n-l} + \frac{dy_{n-k} + ey_{n-s}}{\alpha y_{n-k} + \beta y_{n-s}}, \\ &= ay_n + by_{n-t} + cy_{n-l} + \frac{dy_{n-k}}{\alpha y_{n-k} + \beta y_{n-s}} + \frac{ey_{n-s}}{\alpha y_{n-k} + \beta y_{n-s}}, \\ &\leq ay_n + by_{n-t} + cy_{n-l} + \frac{dy_{n-k}}{\alpha y_{n-k}} + \frac{ey_{n-s}}{\beta y_{n-s}}, \\ &\leq ay_n + by_{n-t} + cy_{n-l} + \frac{d}{\alpha} + \frac{e}{\beta}. \end{aligned}$$

We have $y_{n+1} \leq z_{n+1}$, where $z_{n+1} = az_n + bz_{n-t} + cz_{n-l} + \frac{d}{\alpha} + \frac{e}{\beta}$ linear non-homogenous equation. It is easy to see that the solution of this equation is locally asymptotically stable and converges to the equilibrium point $\bar{z} = \frac{d\beta + \alpha e}{\alpha\beta(1 - (a+b+c))}$ if $a + b + c < 1$. By using the inequality theorem we have that the solution of Eq. (1) is bounded. \square

Theorem 6. Every solution of Eq. (1) is unbounded if $a > 1$ or $b > 1$ or $c > 1$.

Proof. Let $\{y_n\}_{n=-\delta}^{\infty}$ be a solution of Eq. (1). Then from Eq. (1) we see that

$$y_{n+1} = ay_n + by_{n-t} + cy_{n-l} + \frac{dy_{n-k} + ey_{n-s}}{\alpha y_{n-k} + \beta y_{n-s}} > ay_n \text{ for all } n \geq 1.$$

We see that the right hand side can be written as follows

$$z_{n+1} = az_n.$$

then

$$z_n = a^n z_0.$$

and this equation is unstable because $a > 1$, and $\lim_{n \rightarrow \infty} z_n = \infty$. Then by using ratio test $\{y_n\}_{n=-\delta}^{\infty}$ is unbounded from above. Using the same technique, we can prove the other cases. \square

6. Periodic solutions

Here we study the existence of periodic solutions of Eq. (1).

Theorem 7. If t, l, k , are an even and s is an odd then Eq. (1) has a prime period two solutions if and only if

$$(e - d)(\alpha - \beta)(a + b + c + 1) - 4(e\alpha(a + b + c) + d\beta) > 0. \quad (9)$$

Proof. First suppose that there exists a prime period two solution

$$\dots p, q, p, q, \dots,$$

of Eq. (1). If t, l and k are even and s is an odd then $y_n = y_{n-t} = y_{n-l} = y_{n-k}$ and $y_{n+1} = y_{n-s}$. It follows from Eq. (1) that

$$p = aq + bq + cq + \frac{dq + ep}{\alpha q + \beta p} \text{ and } q = ap + bp + cp + \frac{dp + eq}{\alpha p + \beta q}.$$

Therefore,

$$\beta p^2 + \alpha pq = \alpha(a + b + c)q^2 + \beta(a + b + c)pq + dq + ep. \quad (10)$$

and

$$\beta q^2 + \alpha pq = \alpha(a + b + c)p^2 + \beta(a + b + c)pq + dp + eq. \quad (11)$$

By subtracting (11) from (10), we deduce

$$p + q = \frac{e - d}{\beta + \alpha(a + b + c)}. \quad (12)$$

Again, adding (10) and (11), we have

$$pq = \left(\frac{(e - d)(e\alpha(a + b + c) + d\beta)}{(\beta + \alpha(a + b + c))^2} \right) \left(\frac{1}{(\alpha - \beta)(a + b + c + 1)} \right). \quad (13)$$

where $e > d$ and $\alpha > \beta$.

Let p and q are the two positive distinct real roots of the quadratic equation

$$t^2 - (p + q)t + pq = 0.$$

$$\begin{aligned} t^2 - \left(\frac{e - d}{\beta + \alpha(a + b + c)} \right)t \\ + \left(\frac{(e - d)(e\alpha(a + b + c) + d\beta)}{(\beta + \alpha(a + b + c))^2(\alpha - \beta)(a + b + c + 1)} \right) = 0 \end{aligned} \quad (14)$$

Thus, we deduce

$$\begin{aligned} \left(\frac{e - d}{\beta + \alpha(a + b + c)} \right)^2 - 4 \left(\frac{(e - d)(e\alpha(a + b + c) + d\beta)}{(\beta + \alpha(a + b + c))^2} \right) \\ \times \left(\frac{1}{(\alpha - \beta)(a + b + c + 1)} \right) > 0, \end{aligned}$$

or

$$(e - d)(\alpha - \beta)(a + b + c + 1) - 4(e\alpha(a + b + c) + d\beta) > 0.$$

Therefore Inequality (9) holds.

Second suppose that Inequality (9) is true. We will show that Eq. (1) has a prime period two solution.

Suppose that

$$p = \frac{(e - d) + \zeta}{2(\beta + \alpha A)} \text{ and } q = \frac{(e - d) - \zeta}{2(\beta + \alpha A)},$$

where $\zeta = \sqrt{(e - d)^2 - \frac{4(e - d)(e\alpha A + d\beta)}{(\alpha - \beta)(A + 1)}}$ and $A = a + b + c$.

Therefore p and q are distinct real numbers.

Set

$$\begin{aligned} x_{-t} = q, \quad x_{-l} = q, \quad x_{-k} = q, \quad x_{-s} = p, \dots, \quad x_{-3} = p, \quad x_{-2} = q, \\ x_{-1} = p, \quad x_0 = q. \end{aligned}$$

We would like to show that

$$x_1 = x_{-1} = p \text{ and } x_2 = x_0 = q.$$

It follows from Eq. (1) that

$$x_1 = aq + bq + cq + \frac{dq + ep}{\alpha q + \beta p} = Aq + \frac{d \left(\frac{(e - d) - \zeta}{2(\beta + \alpha A)} \right) + e \left(\frac{(e - d) + \zeta}{2(\beta + \alpha A)} \right)}{\alpha \left(\frac{(e - d) - \zeta}{2(\beta + \alpha A)} \right) + \beta \left(\frac{(e - d) + \zeta}{2(\beta + \alpha A)} \right)}.$$

Dividing the denominator and numerator by $2(\beta + \alpha A)$ we get

$$\begin{aligned} x_1 &= Aq + \frac{d((e-d) - \zeta) + e((e-d) + \zeta)}{\alpha((e-d) - \zeta) + \beta((e-d) + \zeta)} \\ &= Aq + \frac{(e+d)(e-d) + (e-d)\zeta}{(\alpha + \beta)(e-d) + (\beta - \alpha)\zeta}. \end{aligned}$$

Multiplying the denominator and numerator of the right side by $(\alpha + \beta)(e-d) - (\beta - \alpha)\zeta$

$$\begin{aligned} x_1 &= Aq + \frac{(e-d)[(e+d) + \zeta][(\alpha + \beta)(e-d) - (\beta - \alpha)\zeta]}{[(\alpha + \beta)(e-d) + (\beta - \alpha)\zeta][(\alpha + \beta)(e-d) - (\beta - \alpha)\zeta]}, \\ &= Aq + \frac{(e-d)[(e+d)(\alpha + \beta)(e-d) + 2(\alpha e - \beta d)\zeta - (\beta - \alpha)\zeta^2]}{(\alpha + \beta)^2(e-d)^2 - (\beta - \alpha)^2\zeta^2}, \\ &= Aq + \frac{(e-d)[(e+d)(\alpha + \beta)(e-d) + 2(\alpha e - \beta d)\zeta - (\beta - \alpha)((e-d)^2 - \frac{4(e-d)(e\alpha A + d\beta)}{(\alpha - \beta)(A+1)})]}{(\alpha + \beta)^2(e-d)^2 - (\beta - \alpha)^2((e-d)^2 - \frac{4(e-d)(e\alpha A + d\beta)}{(\alpha - \beta)(A+1)})}, \\ &= Aq + \frac{(e-d)[2(e-d)(\alpha e + \beta d) + 2(\alpha e - \beta d)\zeta - (\frac{4(e-d)(e\alpha A + d\beta)}{(A+1)})]}{4\alpha\beta(e-d)^2 + (\frac{4(\alpha - \beta)(e-d)(e\alpha A + d\beta)}{(A+1)})}, \\ &= Aq + \frac{2(e-d)[(\alpha e + \beta d)(A+1) - 2(e\alpha A + d\beta)] + 2(A+1)(\alpha e - \beta d)\zeta}{4\alpha\beta(e-d)(A+1) + 4(\alpha - \beta)(e\alpha A + d\beta)}, \\ &= A\left(\frac{(e-d) - \zeta}{2(\beta + \alpha A)}\right) + \frac{(e-d)(\alpha e - \beta d)(1-A) + (A+1)(\alpha e - \beta d)\zeta}{2(\alpha A + \beta)(e\alpha - d\beta)}, \\ &= \frac{(e-d)A - A\zeta + (e-d)(1-A) + (A+1)\zeta}{2(\alpha A + \beta)} = \frac{(e-d) + \zeta}{2(\alpha A + \beta)} = p. \end{aligned}$$

Similarly as before, it is easy to show that

$$x_2 = q.$$

Then by induction we get

$$x_{2n} = q \quad \text{and} \quad x_{2n+1} = p \quad \text{for all} \quad n \geq -\delta.$$

Thus Eq. (1) has the prime period two solution

$$\dots, p, q, p, q, \dots,$$

where p and q are the distinct roots of the quadratic Eq. (14) and the proof is complete. \square

The following Theorem can be proved similarly.

Theorem 8. Eq. (1) has a prime period two solutions if and only if

$$(i) (d-e)(\beta - \alpha)(1+a+b+c) > 4(d\beta(a+b+c) + \alpha e),$$

$$t, l, s - \text{even and } k - \text{odd}.$$

$$(ii) (e-d)(\alpha - \beta)(1+a+c-b) > 4(e\alpha(a+c) + d\beta(1-b)),$$

$$l, k - \text{even and } t, s - \text{odd}.$$

$$(iii) (d-e)(\beta - \alpha)(1+a+c-b) > 4(d\beta(a+c) + e\alpha(1-b)),$$

$$l, s - \text{even and } t, k - \text{odd}.$$

$$(iv) (e-d)(\alpha - \beta)(1+a+b-c) > 4(e\alpha(a+b) + d\beta(1-c)),$$

$$t, k - \text{even and } l, s - \text{odd}.$$

$$(v) (d-e)(\beta - \alpha)(1+a+b-c) > 4(e\alpha(1-c) + d\beta(a+b)),$$

$$l, k - \text{odd and } t, s - \text{even}.$$

$$(vi) (d-e)(\beta - \alpha)(1+a-b-c) > 4(d\beta a + e\alpha(1-b-c)),$$

$$t, l, k - \text{odd and } s - \text{even}.$$

$$(vii) (e-d)(\alpha - \beta)(1+a-b-c) > 4(e\alpha a + d\beta(1-b-c)),$$

$$t, l, s - \text{odd and } k - \text{even}.$$

Theorem 9. If t, l, k , and s are an even and $a+b+c + \frac{d+e}{\alpha+\beta} \neq 1$, then Eq. (1) has no prime period two solutions.

Proof. Suppose that there exists a prime period two solution \dots, p, q, p, q, \dots , of Eq. (1). We see from Eq. (1) when t, l, k and s are an

even then

$$p = aq + bq + cq + \frac{dq + eq}{\alpha q + \beta q}, \quad (15)$$

$$q = ap + bp + cp + \frac{dp + ep}{\alpha p + \beta p}. \quad (16)$$

Subtracting (16) from (15) gives

$$\left(1 - a - b - c - \frac{d+e}{\alpha+\beta}\right)(p-q) = 0,$$

Since $a+b+c + \frac{d+e}{\alpha+\beta} \neq 1$, then $p = q$. This is a contradiction. Thus, the proof is completed. \square

Theorem 10. Eq. (1) has no prime period two solutions if one of the following statements holds

$$(i) 1 + a \neq b + c + \frac{d+e}{\alpha+\beta}, \quad l, k, s, t - \text{odd}.$$

$$(ii) 1 + a + b + c \neq \frac{d+e}{\alpha+\beta}, \quad l, t - \text{even and } k, s - \text{odd}.$$

$$(iii) 1 + a + \frac{d+e}{\alpha+\beta} \neq b + c, \quad k, s - \text{even and } t, l - \text{odd}.$$

$$(iv) 1 + a + c + \frac{d+e}{\alpha+\beta} \neq b, \quad l, k, s - \text{even and } t - \text{odd}.$$

$$(v) 1 + a + b + \frac{d+e}{\alpha+\beta} \neq c, \quad t, k, s - \text{even and } l - \text{odd}.$$

$$(vi) 1 + a + b \neq c + \frac{d+e}{\alpha+\beta}, \quad t - \text{even and } l, k, s - \text{odd}.$$

$$(vii) 1 + a + c \neq b + \frac{d+e}{\alpha+\beta}, \quad l - \text{even and } t, k, s - \text{odd}.$$

Proof. As the proof of the previous Theorem. \square

7. Numerical examples

In this section we present some numerical examples in order to confirm the results of the previous sections and to support our theoretical discussions. These examples represent different types of qualitative behavior of solutions of Eq. (1)

Example 1. Fig. 1 shows that the solution of the difference Eq. (1) is local stability if $t = 5, l = 4, k = 2, s = 3, a = 0.15, b =$

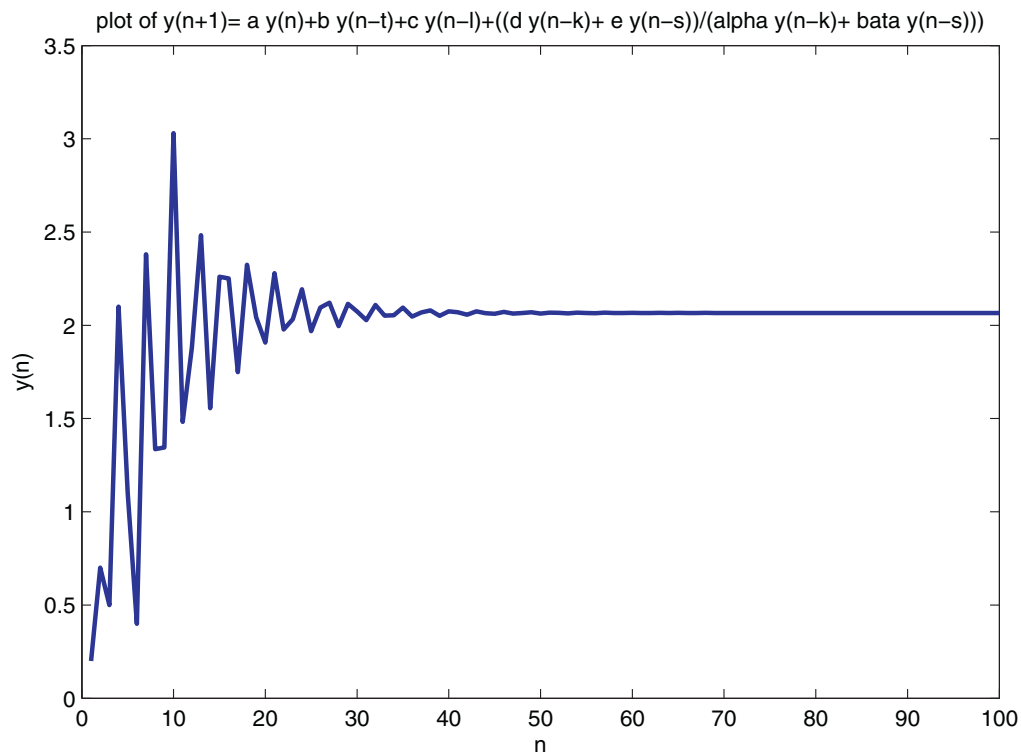


Fig. 1. Draw the behavior of the solution of Eq. (1).

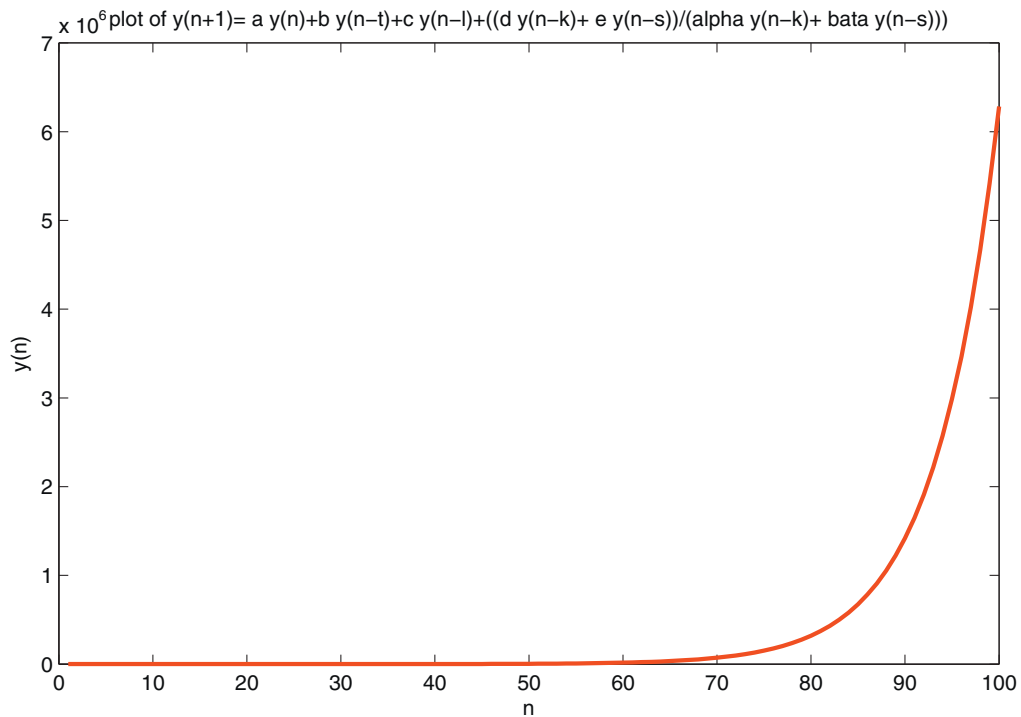


Fig. 2. Sketch the behavior of the solution of Eq. (1).

0.1, $c = 0.2$, $d = 2$, $e = 0.5$, $\alpha = 0.6$, $\beta = 1.6$ and the initial conditions $x_{-5} = 0.2$, $x_{-4} = 0.7$, $x_{-3} = 0.5$, $x_{-2} = 2.1$, $x_{-1} = 1.1$ and $x_0 = 0.4$.

Example 2. See Fig. (2) when we take the difference Eq. (1) with $t = 5$, $l = 4$, $k = 2$, $s = 3$, $a = 0.9$, $b = 0.2$, $c = 0.3$, $d = 2$, $e = 0.5$, $\alpha = 0.6$, $\beta = 1.6$ and the initial conditions $x_{-5} = 0.2$, $x_{-4} = 0.7$,

$x_{-3} = 0.5$, $x_{-2} = 2.1$, $x_{-1} = 1.1$ and $x_0 = 0.4$, the solution is unstable.

Example 3. The solution of the difference Eq. (1) is globally asymptotically stable if $t = 5$, $l = 4$, $k = 2$, $s = 3$, $a = 0.4$, $b = 0.03$, $c = 0.2$, $d = 3$, $e = 0.4$, $\alpha = 0.6$, $\beta = 2.6$ and the initial conditions $x_{-5} = 0.2$, $x_{-4} = 0.7$, $x_{-3} = 0.5$, $x_{-2} = 2.1$, $x_{-1} = 1.1$ and $x_0 = 0.4$ (See Fig. 3).



Fig. 3. Sketch the behavior of the solution of Eq. (1) is global stable when $d\beta > \alpha e$.

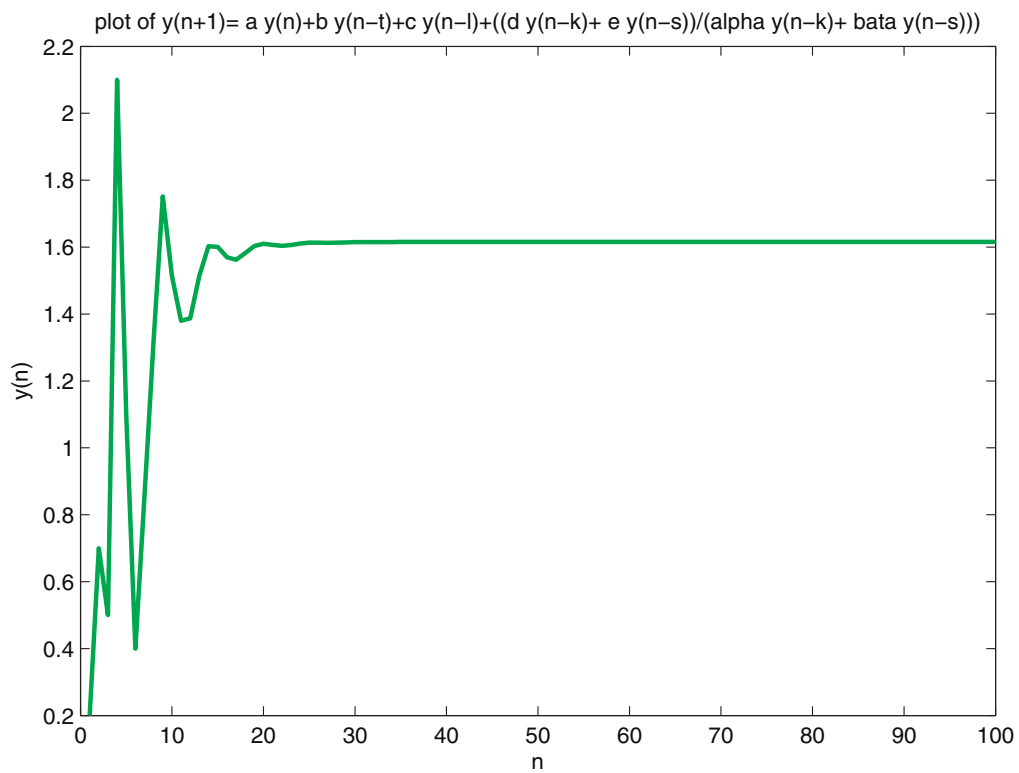


Fig. 4. Plot the behavior of the solution of Eq. (1) is global stable when $\alpha e > d\beta$.

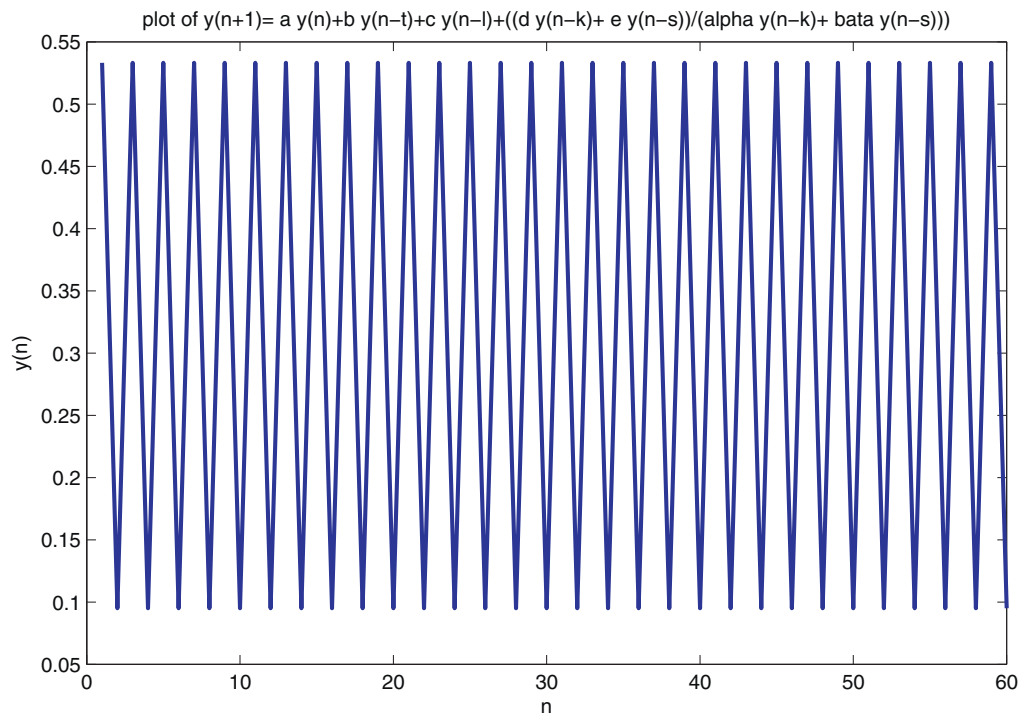


Fig. 5. Plot the periodicity of the solution of Eq. (1).

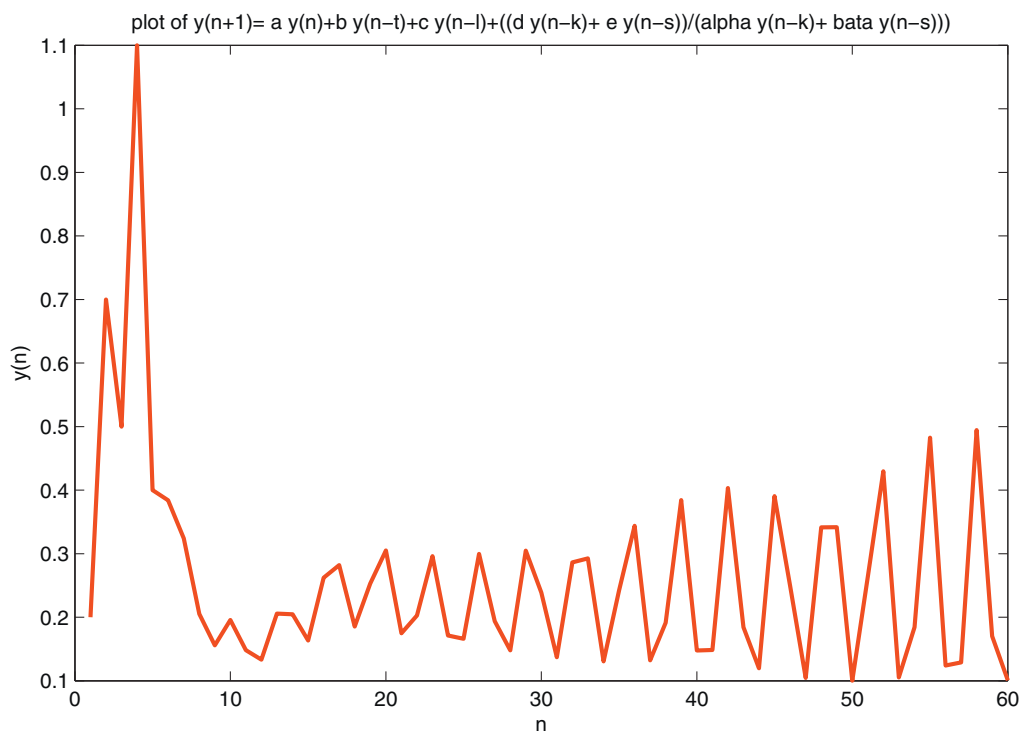


Fig. 6. Draw of the solution of Eq. (1) has no periodic.

Example 4. Fig. 4 shows that the solution of the difference Eq. (1) is globally asymptotically stable if $t = 5$, $l = 4$, $k = 2$, $s = 3$, $a = 0.4$, $b = 0.01$, $c = 0.1$, $d = 1$, $e = 0.9$, $\alpha = 1.8$, $\beta = 0.6$ and the initial conditions $x_{-5} = 0.2$, $x_{-4} = 0.7$, $x_{-3} = 0.5$, $x_{-2} = 2.1$, $x_{-1} = 1.1$ and $x_0 = 0.4$.

Example 5. Fig. 5 shows the solution of Eq. (1) has a prime period two solution $t = l = k = 4$, $s = 5$, $a = 0.03$, $b = 0.04$, $c = 0.02$,

$d = 0.01$, $e = 0.5$, $\alpha = 0.6$, $\beta = 2$ and the initial conditions $x_{-5} = p$, $x_{-4} = q$, $x_{-3} = p$, $x_{-2} = q$, $x_{-1} = p$ and $x_0 = q$.

Example 6. Fig. 6 shows the solution of Eq. (1) has no prime period two solution $t = l = k = 4$, $s = 2$, $a = 0.03$, $b = 0.04$, $c = 0.02$, $d = 0.01$, $e = 0.5$, $\alpha = 0.6$, $\beta = 2$ and the initial conditions $x_{-4} = 0.2$, $x_{-3} = 0.7$, $x_{-2} = 0.5$, $x_{-1} = 1.1$ and $x_0 = 0.4$.

Acknowledgment

The author thanks the editor and the referees for their constructive comments and suggestions which improved the quality of this paper.

References

- [1] R.P. Agarwal, *Difference Equations and Inequalities*, first ed., Marcel Dekker, 1992.
- [2] E.A. Grove, G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman & Hall/CRC, London/Boca Raton, 2005.
- [3] V.L. Kocic, G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.
- [4] M.R.S. Kulenovic, G. Ladas, *Dynamics of Second Order Rational Difference Equations; with Open Problems and Conjectures*, Chapman and Hall/HRC Boca Raton, 2002.
- [5] E. Camouzis, G. Ladas, *Dynamics of Third-Order Rational Difference Equations; With Open Problems and Conjectures*, Chapman and Hall/HRC Boca Raton, 2008.
- [6] H. Sedaghat, *Nonlinear Difference Equations, Theory and Applications to Social Science Models*, Kluwer Academic Publishers, Dordrecht, 2003.
- [7] M.A. Al-Shabi, R. Abo-Zeid, Global asymptotic stability of a higher order difference equation, *Appl. Math. Sci.* 4 (17) (2010) 839–847.
- [8] M.R.S. Kulenovic, G. Ladas, N.P. Prokup, A rational difference equation, *Comp. Math. Appl.* 41 (2001) 671–678.
- [9] C.H. Gibbons, M.R.S. Kulenović, G. Ladas, On the recursive sequence $x_{n+1} = \frac{\alpha + \gamma x_{n-2k+1} + \delta x_{n-2l}}{A + x_{n-2l}}$, *Math. Sci. Res. Hot-Line* 4 (2) (2000) 1–11.
- [10] E. Camouzis, G. Ladas, H.D. Voulfov, On the dynamics of $x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{A + x_{n-2}}$, *J. Differ. Equ. Appl.* 9 (8) (2003) 731–738.
- [11] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \frac{\alpha x_{n-1}}{1 + B x_n x_{n-1}}$, *Appl. Math. Comp.* 156 (2004) 587–590.
- [12] S. Kalabušić, M. Kulenović, On the dynamics of $x_{n+1} = p_n + \frac{x_{n-1}}{x_{n-2}}$, $n = 0, 1, \dots$ with a periodic coefficient, *Comm. Appl. Nonlinear Anal.* 13 (2) (2006) 37–57.
- [13] S. Kalabušić, M.R.S. Kulenović, C.B. Overdeep, Dynamics of the recursive sequence $x_{n+1} = \frac{\beta x_{n-1} + \delta x_{n-k}}{B x_{n-1} + D x_{n-k}}$, *J. Difference Equ. Appl.* 10 (10) (2004) 915–928.
- [14] S. Kalabušić, M.R.S. Kulenović, Rate of convergence of solutions of rational difference equation of second order, *Adv. Difference Equ.* 2004 (2004) 121–139.
- [15] R. Abo-Zeid, Attractivity of two nonlinear third order difference equations, *J. Egypt. Math. Soc.* 21 (3) (2013) 241–247.
- [16] A.E. Hamza, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$, *J. Math. Anal. Appl.* 322 (2006) 668–674.
- [17] M.A. Obaid, E.M. Elsayed, M.M. El-Dessoky, Global attractivity and periodic character of difference equation of order four, *Discrete Dyn. Nat. Soc.* 2012 (2012) 20. Article ID 746738.
- [18] E.M.E. Zayed, Dynamics of the nonlinear rational difference equation $x_{n+1} = a x_n + b x_{n-k} + \frac{p x_n + x_{n-k}}{q + x_{n-k}}$, *Eur. J. Pure Appl. Math.* 3 (2010) 254–268.
- [19] M.A. El-Moneam, On the dynamics of the higher order nonlinear rational difference equation, *Math. Sci. Lett.* 3 (2) (2014) 121–129.
- [20] S.E. Das, Dynamics of a nonlinear rational difference equation, *Hacet. J. Math. Stat.* 42 (1) (2013) 9–14.
- [21] R. Abo-Zeid, On the oscillation of a third order rational difference equation, *J. Egypt. Math. Soc.* 23 (2015) 62–66.